Probabilistic Methods in Combinatorics

Instructor: Oliver Janzer

Assignment 12

To solve for the Example class on 20th May. Submit the solution of Problem 4 by Sunday 18th May if you wish feedback on it. Some hints will be given on Friday 16th May.

Problem 1. Let $G = (V, E)$ be a graph with maximum degree Δ. Show that there exists a colouring of the vertices of G with at most $100\Delta^4$ colours such that, for each vertex v, no two vertices in $N(v)$ have same colour.

Problem 2. Let $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be two sequences over a finite alphabet Σ. A *common subsequence* of A and B is a sequence $(C_1, C_2, \ldots, C_k)$ such that $C_1, \ldots, C_k$ appear in A in order (not necessarily contiguously), and $C_1, \ldots, C_k$ appear in B in order (again, not necessarily contiguously). The *Longest Common Subsequence (LCS)* of A and B is a common subsequence of A and B of maximum possible length. Let $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be two independent uniformly random sequences of length n over the alphabet $\{0, 1\}$. Let L be the length of the LCS of A and B. Show that

$$\mathbb{P}(|L - \mathbb{E}[L]| \geq 100\sqrt{n}) \leq 1/100.$$

Problem 3. Let $G_i = (V, E_i)$ for $i = 1, 2$ be two graphs on same vertex set, and let $e_i = |E_i|$ for $i = 1, 2$. Show that there exists a partition $V = X \cup Y$ such that the number of cross edges (i.e. having exactly one endpoint in X and Y) is at least $e_i/2 - 10\sqrt{e_i}$ for each $i = 1, 2$.

Problem 4. Let $G = (V, E)$ be the graph whose vertices are all 7^n vectors of length n over $\mathbb{Z}_7$, in which two vertices are adjacent if and only if they differ in precisely one coordinate. Let $U \subseteq V$ be a set of 7^{n-1} vertices of G, and let W be the set of all vertices of G whose distance from U exceeds $(c+2)\sqrt{n}$, where $c > 0$ is a constant. Prove that $|W| \leq 7^n \cdot e^{-c^2/2}$.